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On the perturbed equations of motion of aerosol particles in regions of strong curvature of the fluid streamlines

M. FICHMAN, D. PNUELI and C. GUTFINGER
Faculty of Mechanical Engineering Technion, Israel Institute of Technology Haifa 32000, Israel

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Abstract. Asymptotic forms are obtained and solved for the equations of motion of aerosol particles in various flow fields. The flows chosen to be treated include regions of strong curvature of the fluid streamlines, and regions where the initial motion of the particles differ from that of the undisturbed flow. It is shown that regular perturbations cannot yield the correct details of the trajectories of the particles in such regions, and that singular perturbations must be applied. The parameters whose magnitude control the phenomena come out to be the Stokes numbers, the curvature of the fluid streamlines, and the stopping distance of the aerosol articles. In principle this analysis is analogous to the derivation of the classical boundary layer equations. However, the classical boundary layer equations are field equations, which describe the flow in an Eulerian fashion, and therefore apply to a certain domain, geographically fixed in the field. The aerosol equations of motion considered here are Lagrangian, and as the particles move they carry with them the equations whose perturbed form must thus be modified by the fluid flow encountered along the trajectories traced by the particles. The forms assumed by the perturbed equations of motion are, therefore, different in the various cases treated. Analytical solutions are obtained for two cases which until now were handled only numerically, i.e., particle motion in viscous stagnation flow, and injection of particles into a laminar boundary layer on a flat plate. The comparison with numerical solutions shows that this new method gives rather good approximations.

Introduction

The rate of deposition of particles on surfaces is important in aerosol mechanics. Just to quote a few examples: it indicates the settling of particles coming out of chimneys on the neighboring environment; it specifies how clean the air must be in rooms where electronic microchips are manufactured; and it determines whether certain vegetations are successfully pollinated or not.

A significant part of the analysis leading to the evaluation of the rate of deposition is the solution of the equations of motion of the aerosol particles (Gutfinger and Tardos [1]). These equations are Lagrangian in the sense that their dependent variables are the coordinates of the particles involved. The equations contain the velocity of the fluid around the particle and are, therefore, non-linear because this velocity is given in terms of the particle coordinates, i.e., of the dependent variables. This non-linearity causes difficulties in the solution of the equations, and approximations, analogous to the use of the boundary layer equation in classical fluid mechanics, are quite helpful.

A number of asymptotic solutions for particle trajectories in various flow fields are presented in the literature. Michael and Norey [2] considered asymptotic solutions to the flow around a sphere, for large and small Stokes numbers. The comparison was done with the numerical solution of Langmuir and Blodgett [3]. There was obtained the critical value of the Stokes numbers for which no collision occur. The approximation for small Stokes numbers for a particle motion in a flow around a cylinder was also considered by Stechkina, Kirsh and Fuchs [4]. Banks and Kurowski [5] considered a more general case of particle motion in the flow around bodies of various shapes. In all these analyses it was assumed that at small Stokes numbers the particle trajectories deviate only slightly from the fluid streamlines and initially particles move with the flow. This means that, as a first approximation, a particle follows the streamline on which it has started to move, while higher approximations may still yield only small deviations from the initial streamline. In a previous work Fichman, Pnueli and Gutfinger [6] considered aerosol deposition in the vicinity of a stagnation point, where the flow field included a region of strong curvature of the streamlines. They found that in this region even a small particle could cross many streamlines. In such cases the regular asymptotic solution is no longer valid and a singular asymptotic analysis must be used. In that last mentioned work two flow regions had to be considered. The streamlines in one region were practically perpendicular to those in the other, and the two regions were connected by a third, of strong curvature of the streamlines. Although the singular asymptotic analysis used in that work did give good approximations to integral characteristics, such as the overall displacement of a particle from the streamline, the precise trajectory of the particle was not reproduced.

The present study provides a generalization and a modification of the method developed in that previous work. The modified method yields good approximations for particle trajectories in all regions, including those of strong curvature. The object of this investigation is to formulate such approximations, sufficiently general that they may serve in many other practical situations. Such approximate formulations, however, still require the fluid streamlines to have small curvatures in some fairly large region *upstream* of the solid boundaries, Fig. 1. In such situations there are large regions of parallel flow in which the aerosol particles tend to follow the streamlines and the analysis there is relatively simple. As the flow approaches the solid boundary the streamlines become more and more curved, and finally, as the flow proceeds downstream, they again have a small curvature.

The present analysis concentrates on what takes place in the region of strong curvature of the steamlines in the close vicinity of the solid boundary. In general the equations of motion may contain body forces which act on the particles, e.g., electrical forces. For simplicity the present analysis excludes such body forces, but the results may be extended to include them.

Finally it is noted that the Lagrangian character of the equations affects the way in which their perturbed forms are obtained. Since the equations are not field equations but always apply where the particle is located–it is the location of the particle, or, rather, its trajectory, which starts in the region where the streamlines begin to curve, that determines the form of the perturbed equations. In other words, as the particle moves along its trajectory its motion is governed by different forms of the perturbed equations.

Analysis

The equations of motion for spherical aerosol particles in low concentrations such that each particle may still be considered alone are ([1]):

$$\frac{4}{3}\pi r_p^3 \rho_p \frac{d^2\mathbf{x}'}{dt'^2} = \frac{6\pi\mu r_p}{C}\left(\mathbf{U}' - \frac{d\mathbf{x}'}{dt'}\right) - \mathbf{F}' \tag{1}$$

where $\mathbf{x}'$ is the position vector of the location of the particle, r_p is the particle radius and ρ_p is its density, $\mathbf{U}' = (U, V)$ is the fluid velocity and μ is its viscosity, C is the Cunningham correction factor, and $\mathbf{F}'$ is the body force. The apostrophe attached to the variable letters means that they represent dimensional quantities. The body forces, electrical and gravitational, are neglected in the present analysis. The analysis can be generalized to include these body forces if their order of magnitude so demands. Equation (1) is based on the assumption that the Reynolds number is small and that the friction force is the quasi-stationary Stokes force. It is also assumed that the density of the particles is much higher than the density of the gas, and hence the Basset force and the added mass may be neglected. The present analysis considers only spherical particles. For non-spherical particles additional equations for the rotational motion have to be considered, which complicate the problem.

Let a Cartesian coordinate system be defined, with the x' coordinate tangent to the solid surface and the y' coordinate normal to the surface, Fig. 1. The origin of this coordinate system is chosen to coincide with the beginning of the region of strong curvature of the streamlines. A more precise determination of this point is made in each treatment of a particular example, as seen later. Equation (1) is now rewritten in these coordinates,

$$\frac{4}{3}\pi r_p^3 \rho_p \frac{\mathrm{d}^2 x'}{\mathrm{d}t'^2} = \frac{6\pi\mu r_p}{C}\left(U' - \frac{\mathrm{d}x'}{\mathrm{d}t'}\right) \tag{2}$$

$$\frac{4}{3}\pi r_p^3 \rho_p \frac{\mathrm{d}^2 y'}{\mathrm{d}t'^2} = \frac{6\pi\mu r_p}{C}\left(V' - \frac{\mathrm{d}y'}{\mathrm{d}t'}\right) \tag{3}$$

Let the radius of curvature of the steamlines in the region of strong curvature be assumed a constant, r_c'. The value of the curvature radius can be calculated from the equation of the streamline. As seen in Fig. 1, the y'-wise length over which the streamlines lose their y'-component of the velocity is just r_c', which thus may be taken as a fair measure of the size of this region of strong curvature of each individual steamline. Up to the beginning of this region, i.e., up to the location $y' = 0$, the particles are assumed to follow their streamlines. It is of interest now to find the trajectory of a particle as it enters and passes through this zone of strong curvature.

A concept quite helpful here is that of the *stopping distance* of a particle (Friedlander and Johnstone [7]). The stopping distance, s', of a particle initially at the velocity v_{in} is defined as that distance this particle would travel had it been put with the same initial velocity in a stagnant fluid. Since in the zero of strong curvature the fluid loses its y'-component of the velocity, the stopping distance is a fair measure for the magnitude of the y' distance that the particle will travel towards the solid surface. To obtain this stopping distance we seek the exact solution of equation (3), with,

$$V' = 0\,,$$

and with the initial conditions,

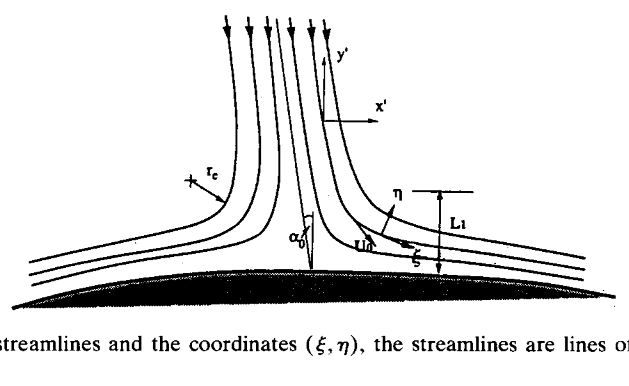

Fig. 1. The streamlines and the coordinates (ξ, η), the streamlines are lines of constant η.

$$y' = 0\,, \qquad \frac{\mathrm{d}y'}{\mathrm{d}t'} = -|\mathbf{U}_0| \quad \text{at } t'/t_0 = 0\,; \tag{4}$$

where t_0 is some representative time, still to be found. The solution of equation (3) under these assumptions is

$$y' = -|\mathbf{U}_0|[1 - \mathrm{e}^{-(t'/t_0)}]\,, \tag{5}$$

with

$$t_0 = \frac{2r_p^2 \rho_p C}{9\mu}\,, \tag{6}$$

and

$$s' = |\mathbf{U}_0| t_0 = \frac{2r_p^2 \rho_p C |\mathbf{U}_0|}{9\mu}\,, \tag{7}$$

and the stopping distance has been obtained. We also note here that for this stopping distance is a lower bound to the actual distance the particle will travel in the negative y' direction, before it loses its y' velocity component.

A characteristic length is needed for scaling. Two possibilities now come to mind. The first one, say L_1, is the distance between the beginning of the region of strong curvature of the streamlines and the solid surface, Fig. 1. It is the length over which the change in the y-component of the velocity takes place. When the curvature of the streamlines is brought about by a flow towards an obstacle-this length is of the sample order of magnitude as the radius of curvature, r'_c. However, when the curvature is caused by the injection of particles in a direction perpendicular to the flow this characteristic length may be quite larger than the radius of curvature.

A second characteristic length, say L_2, may be chosen as the distance over which the change in the x-component of the velocity takes place, which never exceeds half the width of the obstacle to the flow.

In the following analysis the first characteristic length is used, with the possibility $r'_c \ll 1$ considered either as for a case of injection of particles perpendicular to the flow, or as formal asymptotic analysis.

We are now in a position to render all variables and functions dimensionless, using t_0 as the characteristic time, using $|\mathbf{U}_0|$ the flow velocity at $y' = 0$, as the characteristic velocity, and with L used as the characteristic length. Thus,

$$\begin{aligned} &t = t'/t_0, \qquad U = U'/|\mathbf{U}_0|, \qquad V = V'/|\mathbf{U}_0|\,, \\ &r_c = r'_c/L, \qquad x = x'/L, \qquad y = y'/L \end{aligned} \tag{8}$$

and specifically,

$$s = \frac{s'}{L} = \frac{2Cr_p^2 \rho_p |\mathbf{U}_0|}{9\mu L} = St\,. \tag{9}$$

Hence, the dimensionless stopping distance is also the Stokes number of the particle.

The dimensionless form of equations (2) and (3) are

$$St\frac{d^2x}{dt^2} = U(x, y) - \frac{dx}{dt} \tag{10}$$

$$St\frac{d^2y}{dt^2} = V(x, y) - \frac{dy}{dt} \tag{11}$$

We now consider the implications of the magnitudes of the dimensionless r_c, the radius of curvature of the streamlines, and s, the stopping distance of the particle, which also equals St, the Stokes number as defined in equation (9). There are four different cases. The first one is that of $s \gg 1$ when particles hit the surface; the other three cases are for $s \ll 1$, and differ from one another in the ratio between the stopping distance and the radius of curvature of the streamlines. It may be helpful to keep in mind that while r_c is the non-dimensional radius of curvature of the stream line, s may be viewed as the non-dimensional radius of curvature of the particle trajectory where the trajectory is strongly curved.

Case 1: $s \gg 1$

The distance from the beginning of the zone of strong curvature down to the solid surface has been denoted as the characteristic length. Hence, the case of $s \gg 1$ means that the particle always hits the solid surface. It can be shown that in the case, for a surface which is a plane of dimensionless half width $b = b'/L$, b always exceeds one. Thus the particle cannot pass the plane "around the corner".

Case 2: $s \ll 1$, $s/r_c \gg 1$

In this case the stopping distance is very large as compared with the radius of curvature of the fluid steamlines. Hence, the particle passes the high curvature region in the vicinity of the solid boundary, and enters into the region downstream, where the fluid streamlines have again a small curvature. The behavior of the particle in this case is similar to that of a particle injected obliquely into a flow with streamlines of small curvature, Fig. 2. This case is somewhat similar to the classical ones considered, for example, by Michael and Norey [2], Stechkina et al. [4], Banks and Kurowski [5], etc. In their considerations the initial particle velocity equaled the flow velocity while in our case here it does not. This difference is very

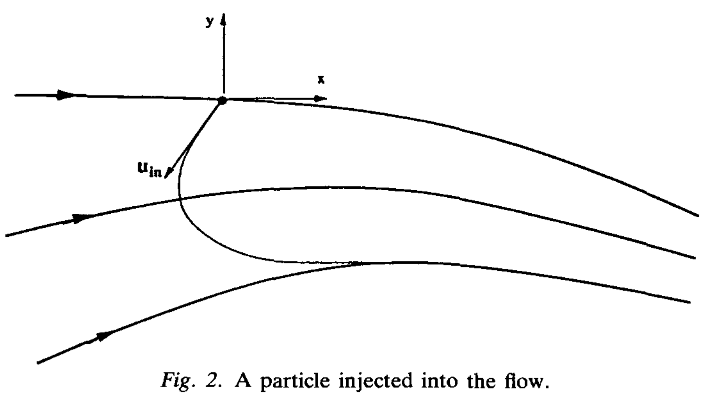

Fig. 2. A particle injected into the flow.

important, as presently shown. The vectorial form of the equations of motion, equations (10) and (11), is

$$St\frac{d^2\mathbf{x}}{dt^2} = \mathbf{U}(\mathbf{x}) - \frac{d\mathbf{x}}{dt} \tag{12}$$

The origin of the coordinate system is at $x = 0$, and the initial conditions are

$$\mathbf{x} = \mathbf{x}_{in} = 0$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}_{in} \quad \text{at } t = 0 \tag{13}$$

The solution is sought as an expansion in powers of the small parameter St,

$$\mathbf{x} = \mathbf{x}^{(0)} + St\,\mathbf{x}^{(1)} + \cdots \tag{14}$$

For the zero approximation one gets

$$\frac{d\mathbf{x}^{(0)}}{dt} = \mathbf{U}(\mathbf{x}^{(0)}) \tag{15}$$

which implies that the particle moves with the fluid and does not leave its streamline. This is a classical and well known result. In our case, however, this solution does not fit the initial conditions of equation (13). In all previous works the initial conditions automatically fit the solution, because they contain the assumption that the particle starts its motion moving with the fluid. We do not assume this and therefore ours is a more general case. Because the zeroth approximation, equation (15), does not satisfy the initial conditions we conclude that there has to be a boundary layer just above the point $y = 0$. Equation (15) yields the outer solution, and the inner solution, inside the boundary layer, must now be sought.

It is noted that the condition $St \ll 1$, when applied to equations (12) and (13) in the form of a regular perturbation, results in equation (15), in which the highest derivative has been lost. This formulation reduces the applicability of our results, e.g., by limiting the classes of initial conditions which can be satisfied. We therefore look for the solution inside the boundary layer using the approach of singular perturbation. We thus define a stretched time, τ, and the inner variable, $\mathbf{X}$

$$\tau = \frac{t}{St} \qquad \mathbf{X} = \frac{\mathbf{x}}{St} \tag{16}$$

Equation (12) now becomes,

$$\frac{d^2\mathbf{X}}{d\tau^2} + \frac{d\mathbf{X}}{d\tau} - \mathbf{U}(St\,\mathbf{X}) = 0 \tag{17}$$

with the initial conditions

$$\mathbf{X} = 0, \frac{d\mathbf{X}}{d\tau} = \mathbf{u}_{in} \quad \text{at } \tau = 0 \tag{18}$$

The inner solution is sought as an expansion in powers of the small parameter St,

$$\mathbf{X} = \mathbf{X}^{(0)} + St \cdot \mathbf{X}^{(1)} + \cdots \tag{19}$$

The fluid velocity field $\mathbf{U} = \mathbf{U}(\mathbf{x})$ is known and is assumed to have a regular expansion near the initial point $\mathbf{x} = 0$,

$$\mathbf{U} = \mathbf{U}(St\,\mathbf{X}) = \mathbf{U}(0) = O(St) \tag{20}$$

This expansion is permissible because $St = s$ and $s/r_c \gg 1$, i.e., the actual deflection zone of each steamline is negligibly small while out of the deflection zone the curvature is small.

Equation (17) for the zeroth approximation is

$$\frac{d^2\mathbf{X}^{(0)}}{d\tau^2} + \frac{d\mathbf{X}^{(0)}}{d\tau} - \mathbf{U}(0) = 0 \tag{21}$$

The solution of equation (21) with the initial condition of equation (18) is

$$\mathbf{X}^{(0)} = [\mathbf{u}_{in} - \mathbf{U}(0)](1 - e^{-\tau}) + \mathbf{U}(0)\tau \tag{22}$$

The inner and outer solutions are combined and the uniformly valid expansion is

$$\mathbf{x} = \mathbf{x}^{(0)} + St[\mathbf{u}_{in} - \mathbf{U}(0)](1 - e^{-t/St}) \tag{23}$$

The first term decreases the streamline which passes the initial point. The second term is the deviation of the particle trajectory from the initial streamline.

Case 3: $s \ll 1$, $s/r_c \approx 1$

In Case 2 ($r_c \ll St \ll 1$) the deflection zone was very small and its size could be neglected. Now the deflection zone is of the order of magnitude of the Stokes number, $r_c \approx St$, and thus cannot be neglected. The initial conditions are still those in equation (13).

The solution for $St \ll 1$, does not satisfy the initial condition. To overcome this, the boundary layer concept must again be introduced, as was done in Case 2. The difference between the two cases is that the velocity components in the present case change very rapidly inside the boundary layer. Therefore, a regular expansion of the fluid velocity components near X_0 cannot represent the flow, as it does in Case 2, and the deflection zone must be considered in more detail.

In the deflection zone the y-component of the fluid velocity changes from its maximal value to almost zero, while the x-component changes from almost zero to a maximum. The change in the absolute velocity, however, is relatively small, i.e., of the order of St. Hence, inside the boundary layer it is helpful to use a set of coordinates changing with the direction of the velocity vector, i.e., curvilinear coordinates connected with the fluid streamlines. Using these coordinates the fluid has only one velocity component, U, that along the streamline. The absolute magnitude of this velocity changes very little.

Let a set of orthogonal curvilinear coordinates connected with the steamlines, Fig. 1, be defined,

$\xi = \xi(x, y)$ measured along the steamlines

$\eta = \eta(x, y)$ measured perpendicular to the streamlines

The splitting streamline is chosen as $\eta = 0$, and $\xi = 0$ is the point of the maximum curvature. Also let a third coordinate, ζ perpendicular to the (ξ, η) plane, be defined. In these coordinates the fluid velocity is

$$U_\xi = U_\xi(\xi, \eta) = U, U_\eta = 0 \tag{24}$$

and the particle velocity is

$$\mathbf{u} = \mathbf{e}_\xi u + \mathbf{e}_\eta v \tag{25}$$

where u and v are its velocity components parallel and perpendicular to the streamline, respectively.

The victorial equations of particle motion, in Eulerian coordinates which do not depend on the coordinate system used, are,

$$St\left[(\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla\left(\frac{u^2}{2}\right)\right] = \mathbf{U} - \mathbf{u} \tag{26}$$

Here, however, only flows in the (ξ, η) plane are considered, i.e., flows that are either two-dimensional or axisymmetric. For these cases the only non-vanishing component of the curl of the velocity is the one perpendicular to the (ξ, η) plane, namely, the ζ component, i.e.,

$$\begin{aligned} &(\nabla \times \mathbf{u})_\xi = 0,\ (\nabla \times \mathbf{u})_\eta = 0\,, \\ &(\nabla \times \mathbf{u})_\zeta = \frac{1}{h_1 h_2}\left[\frac{\partial(vh_2)}{\partial \xi} - \frac{\partial(uh_1)}{\partial h}\right] \end{aligned} \tag{27}$$

where h_1 and h_2 are the Lamé coefficients in the ξ and $d\eta$ directions, respectively.

Equations (27) are substituted into equation (26) and the equations of particle motion become

$$St\left[h_2 u \frac{\partial u}{\partial \xi} - v^2 \frac{\partial h_2}{\partial \xi} + v \frac{\partial(uh_1)}{\partial \eta}\right] = h_1 h_2 (U - u) \tag{28}$$

$$St\left[h_1 u \frac{\partial v}{\partial \eta} - u^2 \frac{\partial h_1}{\partial \eta} + u \frac{\partial(vh_2)}{\partial \xi}\right] = -h_1 h_2 v \tag{29}$$

For $St \ll 1$ equations (28) and (29) reduce to the simple outer solution:

$$u = U, v = 0 \tag{30}$$

For the inner region the inner variables are defined as

$$\begin{aligned} &\bar{\xi} = \xi / St \\ &\bar{\eta} = \eta \end{aligned} \tag{31}$$

The equations of particle motion, equations (28) and (29), in the inner variables become:

$$h_2 u \frac{\partial u}{\partial \bar{\xi}} - v^2 \frac{\partial h}{\partial \bar{\xi}} + St\, v \frac{\partial(uh_1)}{\partial \bar{\eta}} = h_1 h_2 (U - u) \tag{32}$$

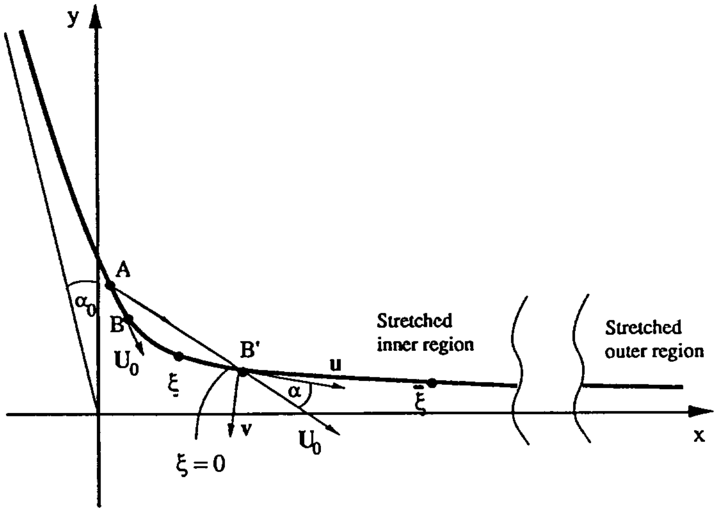

Fig. 3. Stretched coordinates.

$$St\left(h_1 v \frac{\partial v}{\partial \bar{h}} - u^2 \frac{\partial h_1}{\partial \bar{\eta}}\right) + u \frac{\partial (v h_2)}{\partial \bar{\xi}} = -h_1 h_2 v \tag{33}$$

In the inner variables the deflection zone still exists, as shown in Fig. 3, but may be ignored, as a regular perturbation. In fact, the inner layer solution must be matched to the outer solution for $\bar{\xi} \to \pm\infty$. But it can be shown that the $\xi < 0$ there is just a trivial solution which is equivalent to that in the outer region, i.e. the particle moves on the initial streamline. The reason for this is the fact that the mathematical problem is not a two-point boundary value problem but an initial value one, which is ill-defined. Hence, the ordinary procedure which has been used for the 'corner layer' (Kevorkian and Cole [8]) cannot be used here. The suggested approach is similar to that which has been used in the boundary layer problems. The initial point of the boundary layer is chosen at the beginning of the deflection region. The position of this point cannot be determined exactly. But because the width of the boundary layer is of the order of St the influence of the position of the initial point has no significant influence and it is taken at $\xi = -St/2$. in the inner coordinates the deflection region can be neglected without causing any singularity. In Fig. 3 this region is represented by AB'. In the regular coordinates this neglected region shrinks to the region AB, which is of order St^2. The velocity vector at the initial point A is approximately directed towards point B. In the inner coordinates point B transforms to point B' at the end of the deflection region. The velocity vector then changes its direction at point B'. This velocity vector yields the initial condition for the particle velocity in the inner region when the deflection region in the inner coordinates is neglected. Therefore, the initial conditions are

$$u(\bar{\xi}_{in}, \bar{\eta}_{in}) = U_{in} \cos\alpha, \; v(\bar{\xi}_{in}, \bar{\eta}_{in}) = -U_{in} \sin\alpha \tag{34}$$

The value of the angle α is approximately $\pi/4$ in the case when the outer region steamlines are parallel to the x-coordinate. In other situations $\alpha = \pi/4 - \alpha_0/2$, where α_0 is the inclination angle of the tangent to the streamline at the beginning of the outer region, Fig. 3.

The value of α_0 is calculated for every specific case. When the Stokes number does not equal the length of the deflection region, i.e., it differs from 1, the angle α is calculated as

$$\alpha = \frac{s}{r_c}\left(\frac{\pi}{4} - \frac{\alpha_0}{2}\right) \tag{35}$$

These initial conditions are similar to those of Case 2, but here they are in the inner region. Here the particles may be considered as being injected into the fluid stream represented in the stretched coordinates.

The solution of equations (32) and (33) with the initial conditions of equation (34) yields the velocity components u and v, which depend on the independent variables and on the Stokes number.

The inner solution is sought as an asymptotic expansion in powers of St:

$$\begin{aligned} u &= u(St, \xi, \eta) = u(St, \bar{\xi}, \bar{\eta}) = u^{(0)}(\bar{\xi}, \bar{\eta}) + St\, u^{(1)}(\bar{\xi}, \bar{\eta}) + \cdots \\ v &= v(St, \xi, \eta) = v(St, \bar{\xi}, \bar{\eta}) = v^{(0)}(\bar{\xi}, \bar{\eta}) + St\, v^{(1)}(\bar{\xi}, \bar{\eta}) + \cdots \end{aligned} \tag{36}$$

The known functions U, h_1, and h_2 are given by the flow field and do not depend on the Stokes number. At this point we are going to assume the U, h_1, and h_2 have regular expansions near the point $\xi = 0$. This assumption is justified by noting that within each individual subregion all three functions change rather slowly, and regularly. In that part of the analysis where the inner coordinates are used, the streamlines pass through the region of strong curvature and then straighten up. A doubt may arise as to whether regular expansions can cover up this combination. But then we note that in this particular part of the analysis the region of strong curvature is altogether ignored. We therefore assume that the three functions do have regular expansions near $\xi = 0$, for $|\eta| > \varepsilon$, where ε is a small fixed number. These expansion have the forms:

$$\begin{aligned} U &= U(\xi, \eta) = U^{(0)}(0, \eta) + \xi U^{(1)}(0, \eta) + \cdots = U^{(0)}(0, \bar{\eta}) + St\, \bar{\xi} U^{(1)}(0, \bar{\eta}) + \cdots \\ h_1 &= h_1(\xi, \eta) = h_1^{(0)}(0, \eta) + \xi h_1^{(1)}(0, \eta) + \cdots = h_1^{(0)}(0, \bar{\eta}) + St\, \bar{\xi} h_1^{(1)}(0, \bar{\eta}) + \cdots \\ h_2 &= h_2(\xi, \eta) = h_2^{(0)}(0, \eta) + \xi h_2^{(1)}(0, \eta) + \cdots = h_2^{(0)}(0, \bar{\eta}) + St\, \bar{\xi} h_2^{(1)}(0, \bar{\eta}) + \cdots \end{aligned} \tag{37}$$

The zeroth approximation of equations (32) and (33) comes out as

$$u^{(0)} \frac{\partial u^{(0)}}{\partial \bar{\xi}} = h_1^{(0)}(U^{(0)} - u^{(0)}) \tag{38}$$

$$\frac{\partial v^{(0)}}{\partial \bar{\xi}} + \frac{h_1^{(0)}}{u^{(0)}} v^{(0)} = 0 \tag{39}$$

The solution of equations (38), (39) together with the initial conditions given by equation (36) is

$$u^{(0)} - U_{in} \cos \alpha + U^{(0)} \ln\left(\frac{u^{(0)} - U^{(0)}}{U_{in} \cos \alpha - U^{(0)}}\right) = -h_1^{(0)}(\bar{\xi} - \bar{\xi}_{in}) \tag{40}$$

$$v^{(0)} = \frac{U_{in} \sin \alpha}{U^{(0)} - U_{in} \cos \alpha} (u^{(0)} - U^{(0)}) \tag{41}$$

We are looking for the particle trajectory $\bar{\xi} = \bar{\xi}(\tau)$ where τ is the inner time. The outer time t is related to τ by

$$\tau = t/St \tag{42}$$

The change of the $\bar{\eta}$-coordinate in the inner layer is of the same order as that of $\bar{\xi} \sim St$. Thus

$$\bar{\eta} = \bar{\eta}_{in} + O(St) \tag{43}$$

The fluid velocity U_0 and the Lame coefficients may be expanded as

$$\begin{aligned} U^{(0)}(\bar{\eta}) &= U^{(00)}(\bar{\eta}_{in}) + o(St) = 1 + o(St) \\ h_1^{(0)}(\bar{\eta}) &= h_1^{(0)}(\bar{\eta}_{in}) + o(St) = h_1^{(00)} + o(St) \\ h_2^{(0)}(\bar{\eta}) &= h_2^{(0)}(\bar{\eta}_{in}) + o(St) = h_2^{(00)} + o(St) \end{aligned} \tag{44}$$

Now, we define a new parameter $\gamma = u^{(0)}/U^{(00)}$. Using γ together with the zeroth approximation of the expansion, equation (44), the solution of equation (39) is:

$$\bar{\xi}\frac{h_1^{(00)}}{U^{(00)}} = -\ln\left(\frac{\gamma - 1}{\frac{U_{in}}{U^{(00)}}\cos\alpha - 1}\right) - \gamma + \frac{U_{in}}{U^{(00)}}\cos\alpha + \bar{\xi}_{in}\frac{h_1^{(00)}}{U^{(00)}} \tag{45}$$

For the velocity components

$$\begin{aligned} u^{(0)} &= h_1^{(0)}\frac{\mathrm{d}\xi}{\mathrm{d}t} \approx h_1^{(00)}\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}\tau}, \\ v^{(0)} &\approx h_2^{(0)}\frac{\mathrm{d}\eta}{\mathrm{d}t} \approx h_2^{(00)}\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}\tau} \end{aligned} \tag{46}$$

Substitution of equation (46) into equation (41) and their integration yield

$$h_2^{(00)}(\eta - \eta_{in}) = St\frac{U_{in}\sin\alpha}{\frac{U_{in}}{U^{(00)}}\cos\alpha - 1}\left(\gamma - \frac{U_{in}}{U^{(00)}}\cos\alpha\right) \tag{47}$$

with η_0 is the coordinate at $t = 0$, where the particle enters the inner layer. Equations (45) and (47) give the parametric representation of the particle trajectory with γ as a parameter.

In particle deposition theory it is of interest to find the overall change of the coordinate η, which determines the collection efficiency. Let η_∞ be the coordinate at the exit from the deflection zone; then for $\eta \rightarrow \eta_\infty$ and $\xi \rightarrow \infty$, $\gamma = u^{(0)}/U^{(0)} = 1$, and equation (47) leads to

$$\Delta\eta = \eta_{in} - \eta_\infty = \frac{St\, U^{(00)}}{h_2^{(00)}}\sin\alpha \tag{48}$$

Thus, the overall change $\Delta\eta$ is obtained for the various flow fields.

The characteristic velocity in the Stokes numbers given in equation (48) is local velocity at the maximum curvature point of the streamline. This velocity is the one relevant for the deposition in stagnation flow. Data in the literature, however, are reported in terms of

Stokes number based on the main stream velocity U_∞. To facilitate comparison equation (48) is rewritten in terms of St_∞, to read

$$\Delta\eta = \frac{U^{(00)}}{U_\infty}\frac{St_\infty}{h_2^{(00)}} \tag{49}$$

Values of $U^{(00)}$ and $h_2^{(00)}$ are calculated in each flow field at the point of the maximum curvature of the streamlines. In the coordinate system connected with the streamlines, the stream function depends on the coordinate η only.

Case 4: The trajectories of particles for $s \ll 1$ $s/r_c \ll 1$

This case results in small deviations of the particle trajectory from the streamline. It has been treated by other authors using *regular* perturbation analysis. Here we shortly described the approach of Fichman et al. [5].

In the vicinity of the stagnation point $r_c \ll 1$. This condition allows the representation of a streamline in the deflection zone as a quarter circle with its radius equal to r_{min}, and with the flow velocity $U = U^{(00)}$. The condition $St \ll r_c$ implies that in the deflection zone the particle departs only very slightly from the streamline. Thus the streamlines near the streamline $\eta = \eta_0$ may be used as a polar coordinate system, i.e., $\xi = \alpha$; $\eta = \rho$ where α and ρ and the polar coordinates with the pole in the center of curvature.

The solution is sought in the form of a power series in Stokes number and is according to [5]

$$\rho - r = St\, u^{(00)}\alpha \tag{50}$$

At the exit from the deflection region the particle still has a velocity component perpendicular to the streamline,

$$v_0 = \frac{2U^{(00)^2} St}{2r + \pi\, St\, U_\infty} \tag{51}$$

The particle velocity in the direction of the flow is equal to the flow velocity. Hence, outside the deflection zone the particle may still be considered as moving in a unidirectional flow, similar to that considered in Case 2.

To calculate the overall change in the η-coordinate the changes inside and outside the deflection zone must be combined. Outside the deflection zone

$$|\Delta x| = St\, v_0\,, \tag{52}$$

while inside the deflection region the difference Δr is obtained from equation (52) with $\alpha = \pi/2$. Thus the overall deflection is

$$|\Delta x|_{\text{overall}} = |\Delta x| + |\Delta r| = \left(v_0 + \frac{\pi}{2}U_\infty\right)St \tag{53}$$

Example 1: Particle trajectory in two-dimensional viscous stagnation flow

The motion of a small rigid particle in two-dimensional viscous stagnation flow is considered as an example of the use of the present asymptotic analysis. Following Schlichting [9] the flow field for very small distances from the stagnation flow may be approximated by

$$U = 2xy$$
$$V = y^2 \tag{54}$$

and the stream function is

$$\psi = xy^2$$

The analysis of Case 3 for $s/r \approx 1$ is applied here. The curvilinear coordinates connected with the streamlines are chosen as

$$\eta = xy^2$$
$$\xi = x^2 - \frac{1}{2}y^2 \tag{56}$$

The point of maximum curvature is obtained from the minimum value of the curvature radius of the curve which is described by equation (56)

$$x_0 = 5^{-1/3}\eta^{1/3}, \qquad y_0 = 5^{1/6}\eta^{1/3} \tag{57}$$

The flow velocity at this point is found by combining equation (57) with equation (54)

$$U_0 = |\sqrt{U^2 + V^2}| = \frac{3\eta^{2/3}}{5^{1/6}} \tag{58}$$

The Lamè coefficients for these streamlines are

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} = (4x^2 + y^2)^{-1/2}$$
$$h_2 = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2} = y(4x^2 + y^2)^{-1/2} \tag{59}$$

which at the point of maximum curvature (x_0, y_0) attain the values

$$h_1 = \frac{5^{1/3}}{3}\eta_{in}^{-1/3}, \qquad h_2 = \frac{5^{1/3}}{3}\eta_{in}^{-2/3} \tag{60}$$

In this example we consider $\eta_{in} = 0.1$ and several values of Stokes numbers up to $St \sim 1$, which satisfy the conditions for Case 3. The solution is started at the point $\xi = -St/2$. We assume that a particle coming out of the boundary layer moves on the streamline, and its initial velocity is equal to the flow velocity. The angle α is calculated from equation (35) with $\alpha_0 = 0$. The parameters u_0, h_1, and h_2, at the point of maximum curvature, are taken from equations (58) and (60), respectively, and the particle trajectory is calculated from equations (45) and (47) in terms of ξ and η.

The present singular asymptotic analysis is compared with the regular asymptotic solution for $St \ll 1$, which is obtained by the solution of equations (10), (11) assuming

$$u = \frac{dx}{dt} = u_0 + St\, u_1 + \cdots$$
$$v = \frac{dy}{dt} = v_0 + St\, v_1 + \cdots \tag{61}$$

where the zero approximation is just

$$u_0 = U, \qquad v_0 = V \tag{62}$$

Substitution of this solution into equation (61) yields

$$u = U + St\, u_1, \qquad v = V + St\, v_1 \tag{63}$$

Substitution in equations (10)–(11) yields

$$u_1 = -St\frac{dU}{dt}, \qquad v_1 = -St\frac{dV}{dt} \tag{64}$$

Hence from equation (61):

$$u = -\frac{y^2}{1 - 2y\, St}$$

$$v = 2xy\frac{1 - y\, St}{1 - 4y^2\, St^2} \tag{65}$$

To find the trajectories of the particles equation (65) is integrated numerically using a fourth-order Runge–Kutta scheme. Equation (64) indicates that the solution is valid for $St < 0.5/y$.

The comparison between the singular and the regular asymptotic solutions is done on the basis of the numerical solution of the full equations of particle motion. The full equations, equations (10) and (11), are solved using the fourth order Runge–Kutta scheme. The results are presented in Figs. 4–8. The regular asymptotic solution fails at $St > 0.5/y$. When the Stokes number is very small the boundary layer is also very small; hence the boundary layer

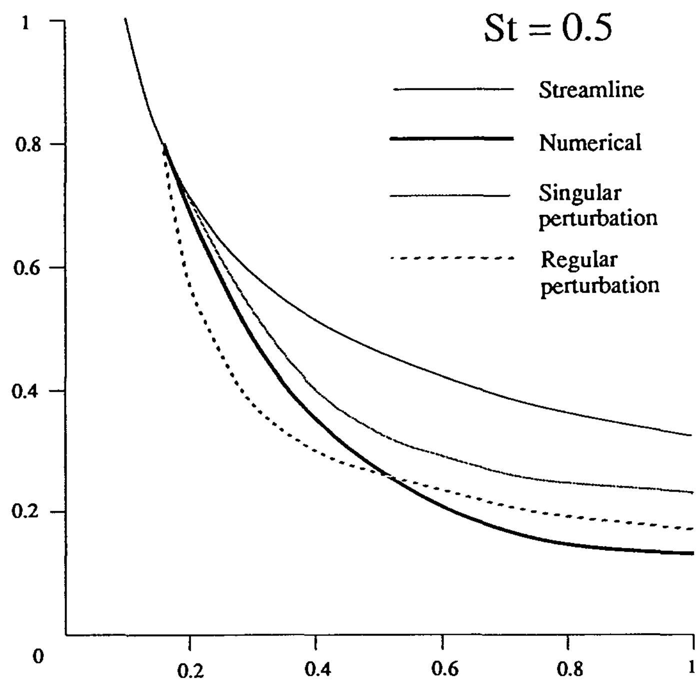

Fig. 4. Particle trajectory in viscous stagnation flow for $St = 0.5$.

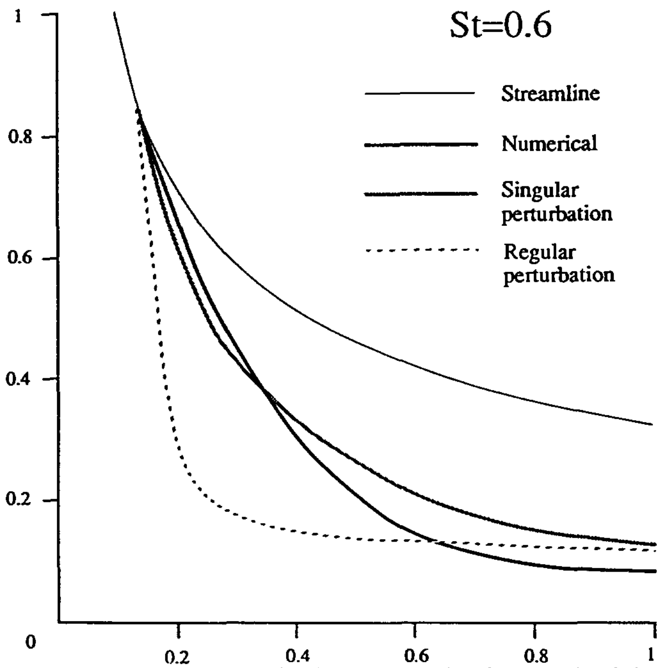

Fig. 5. Particle trajectory in viscous stagnation flow for $St = 0.6$.

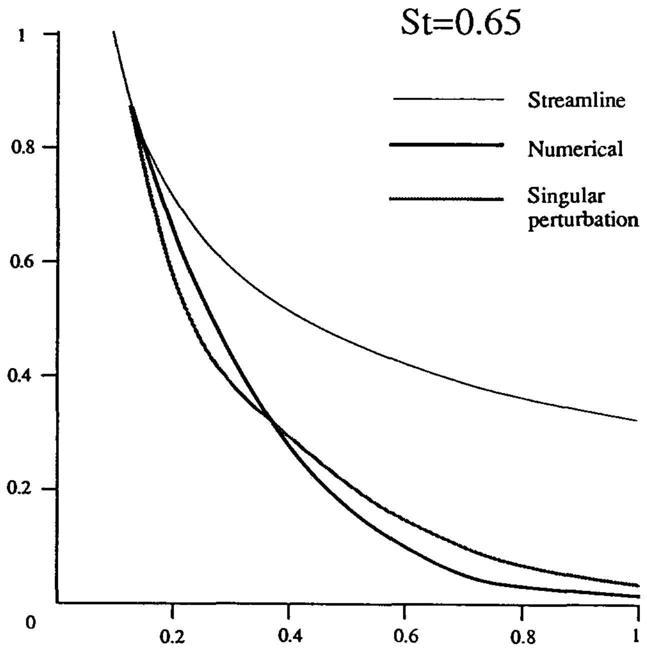

Fig. 6. Particle trajectory in viscous stagnation flow for $St = 0.65$.

constitutes only a small part of the deflection zone. In this case the present solution is not very interesting, and the regular asymptotic solution is valid in all regions up to $St \sim 0.5$. For larger Stokes numbers we first consider Figs. 4 and 5, which show the difference between results obtained by regular ones. Then, Figs. 6 to 8 show results obtained by singular perturbations only, because regular perturbations cannot yield any results in this range,

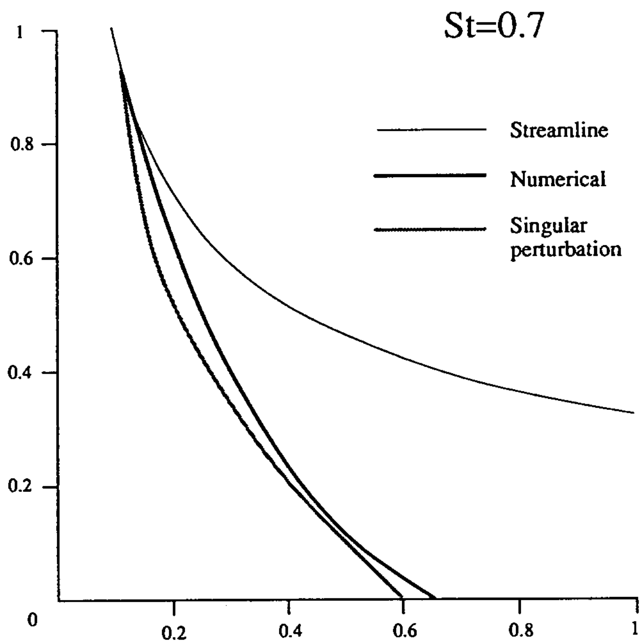

Fig. 7. Particle trajectory in viscous stagnation flow for $St = 0.7$.

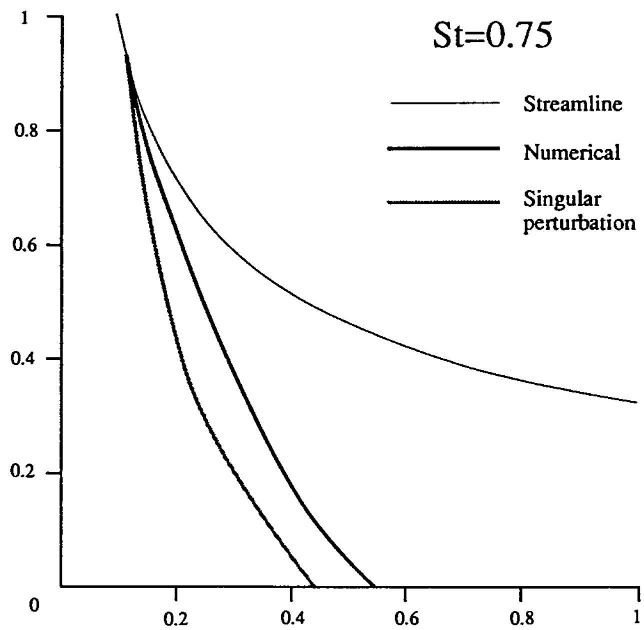

Fig. 8. Particle trajectory in viscous stagnation flow for $St = 0.75$.

equation (65). At Stokes numbers close to 0.7 the trajectories of the particles deviate considerably from the fluid streamlines, and the assumption of small Stokes numbers is not valid at all. Under these conditions the particles leave the streamlines and hit the surface. Still, as shown in Figs. 7 and 8, a good approximation to the complete numerical solution is obtained even in these cases.

Example 2: Particle trajectory in laminar boundary layer over a flat plate

The motion of a particle injected into a laminar boundary layer is interesting in sedimentation on surfaces from contamination sources, injection of tracers into flows, reentrainment of particles from surfaces, etc. There are no analytical solutions in the literature for trajectories of particles injected into a boundary layer, except for the case when the particle initially move with the flow. We now calculate the trajectories of particles injected at various angles into the boundary layer flow, using the solution obtained in Case 2. We note that boundary layer flows do not possess streamlines with strong curvatures. However, the steep angles of injection of the particles cause the particles motion to curve strongly with respect to their preinjection trajectories, and the present mathematics apply.

The flow field in laminar boundary layer on a plate is given by (Schlichting [9])

$$U = U_\infty f'(\eta)$$
$$V = \frac{1}{2}\sqrt{\frac{vU_\infty}{x}}(\eta f' - f) \tag{66}$$

where the function $f(\eta)$ is tabulated in terms of $\eta = y\sqrt{U_\infty / vx}$.

The particle is located initially at some point $\mathbf{x}_0$ with an initial velocity $\boldsymbol{v}_{in}$, which differs from the velocity of the fluid at the same point. The trajectory of the particle is given by equation (23) where $\mathbf{x}^{(0)}$ is the fluid streamline that passes through point $\mathbf{x}_0$ and, hence, $\mathbf{x}^{(0)}$ is connected to the flow velocity by

$$\frac{\mathrm{d}x^{(0)}}{\mathrm{d}t} = U, \qquad \frac{\mathrm{d}y^{(0)}}{\mathrm{d}t} = V \tag{67}$$

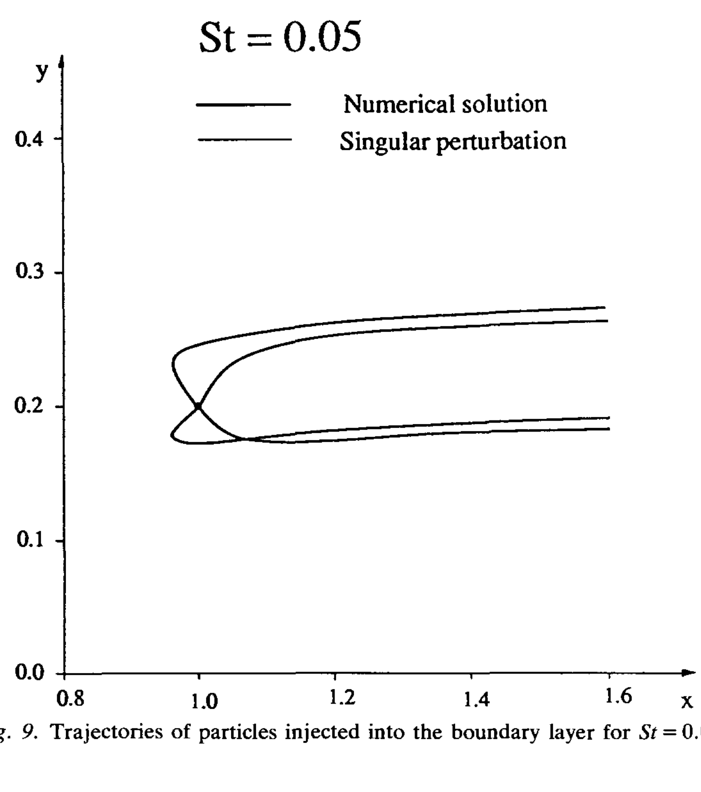

Fig. 9. Trajectories of particles injected into the boundary layer for $St = 0.05$.

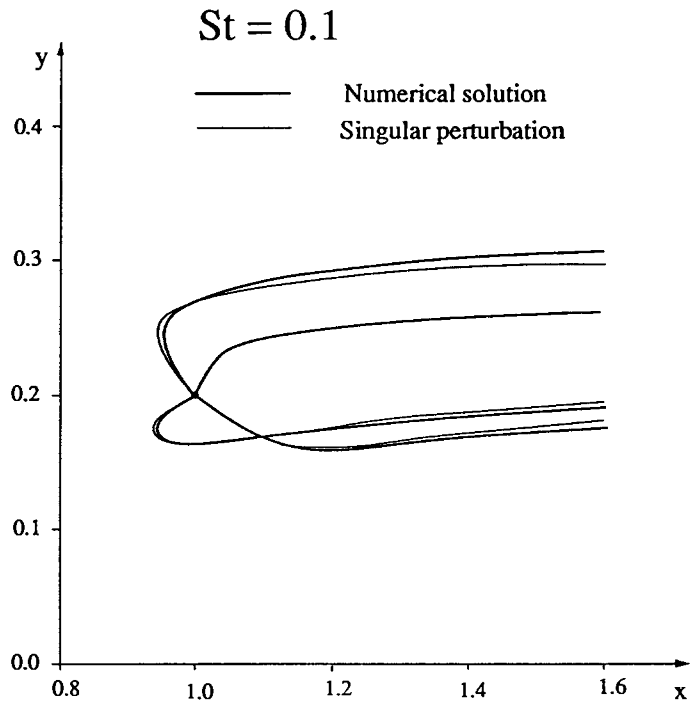

Fig. 10. Trajectories of particles injected into the boundary layer for $St = 0.1$.

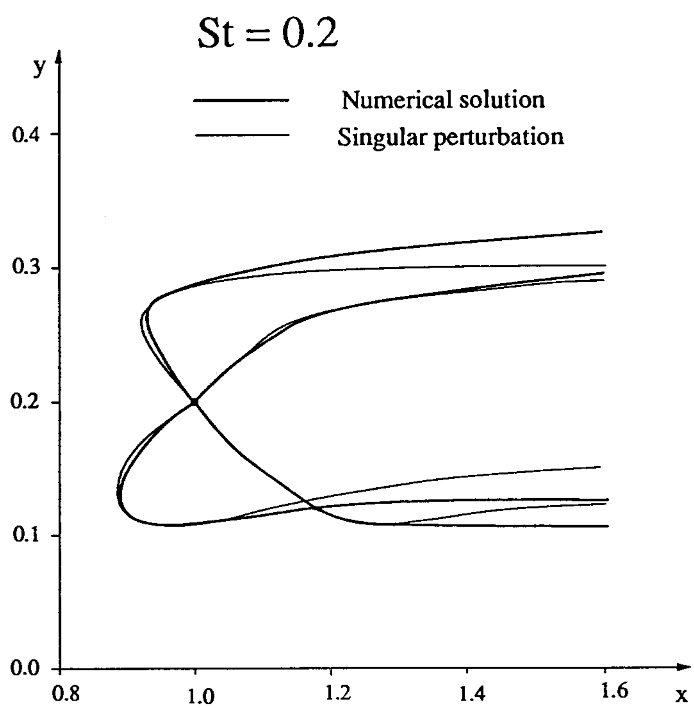

Fig. 11. Trajectories of particles injected into the boundary layer for $St = 0.2$.

The solution of $\mathbf{x}^{(0)}$ is obtained numerically from equations (66) and (67). The present solution is compared with numerical solution of the full equations of particle motion. The full equations, equation (10) and (11) are solved using the fourth order Runge–Kutta scheme. The particle trajectories for various inclined angles are presented in Figs. 9–11. For all cases considered the particles have the same initial absolute velocity and different injection angles. Three different Stokes numbers are considered. The comparison shows a good approximation to the numerical solution for $St < 2$. For $St > 2$. The present solution deviates from the numerical one at some distance from the intial point.

Summary

Asymptotic perturbed forms have been obtained and solved for the equations of motion of aerosol particles in various flow fields. The flows chosen to be treated include regions of strong curvature of the flow streamlines, and initial conditions in regions where the motion of the particles differ from that of the undisturbed flow. It is shown that regular perturbations cannot yield the correct details of the trajectories of the particles in such regions, and that singular perturbations must be applied.

The parameters whose magnitude control the phenomena come out to be the Stokes number, the curvature of the flow streamlines, and the so-called stopping distance of the aerosol particles. Their magnitudes are used to classify the four cases treated in detail in this presentation.

In principle this analysis is analogous to the derivation of the classical boundary layer equations. However, the classical boundary layer equations are field equations, which describe the flow in an Eulerian fashion and therefore apply to a certain domain, geographically fixed in the field. On the other hand, the aerosol equations of motion are Lagrangian; therefore, as the particle moves it carries with it the equations whose perturbed form must thus be modified by the flow encountered along the path traced by the particle. The forms assumed by the perturbed equations of motion are hence different in the various cases treated. Still, the same analysis, based on singular perturbations, leads to solutions which cannot be affected using regular perturbation methods. Quantitative results obtained by the method presented here for particle trajectories in a viscous stagnation flow compare rather well with results obtained by other methods. Another example treated quantitatively is that of the trajectories of particles injected into a viscous boundary layer flow.

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References

1. C. Gutfinger, and G. Tardos, Theoretical and experimental studies on granular bed filtration. *Atmos. Environ.* 13 (1979) 853–867.
2. D.H. Michael and P.W. Norey, Particle collision efficiencies for a sphere. *J. Fluid Mech.* 3 (1969) 565–575.
3. I. Langmuir and K. Blodgett, U.S. Army Air Force Technical Report no. 5418 (1946).

4. I.B. Stechkina, A.A. Kirsh and N.A. Fuchs, Studies on fibrous aerosol filters, IV. Calculation of aerosol deposition in model filters in the range of maximum penetration. *Ann. Occup. Hyg.* 12(i) (1969) 1–8.
5. D.O. Banks and G.J. Kurowski, A perturbation model for the computation of collector efficiency for axially symmetric collectors. *Aerosol Sci. Technol.* 3X (1984) 327–334.
6. M. Fichman, D. Pnueli and C. Gutfinger, Aerosol deposition in the vicinity of a stagnation point. *Aerosol Sci. Technol.* 13 (1990) 281–296.
7. S.K. Friedlander and H.T. Johnstone, Deposition of suspended particles from turbulent gas streams. *Ind. Engng. Chem.* 49 (1957) 1151–1156.
8. J. Kevorkian and J.D. Cole, *Perturbation Methods in Applied Mathematics*. Springer-Verlag, New York Heidelberg Berlin (1981).
9. H. Schlichting, *Boundary Layer Theory*. 7th Ed., McGraw Hill Co., New York, (1979).